Boarding A Plane

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1 Problem statement

One hundred airplane passengers wait in line to board their fully-booked airplane. Each passenger has a unique assigned seat, and boards the plane as follows: If his assigned seat is available, he will sit in it. If, for some reason, someone else occupies his seat, he will avoid a confrontation and choose an unoccupied seat at random (displacing some other future passenger). If the passengers all sit in their assigned seats, then no one will be displaced and everyone will be happy. But the passenger at the front of the line is a troublemaker, and instead of simply sitting in his assigned seat, picks one of the 100 seats at random.

You are at the back of the line. What are the chances that you'll get your assigned seat?

2 Solution

The probability that you'll get your assigned seat is $\frac{1}{2}$. In fact, this result holds for any number of passengers and seats (e.g., 50 passengers boarding a 50-seater plane). That is, for any N-passenger instance of the problem, you get your own seat half the time. We prove this with *strong induction*; that is, we prove it by proving

1. With N=2 passengers, the troublemaker in front and you in back, the probability that you get your assigned seat is $\frac{1}{2}$.
2. If the probability is $\frac{1}{2}$ for a k-passenger instance of the problem (where $k \in \{2, 3, \ldots, N-1\}$), then it is also $\frac{1}{2}$ for N passengers.

The first claim is proved by inspection: With only 2 seats available, there is a $\frac{1}{2}$ chance that the troublemaker will choose his own seat, allowing you to take your own seat. The other half of the time, the troublemaker takes *your* seat, forcing you out of it.

Now we prove the second claim. Let the N passengers in line be numbered from 1 to N, with 1 being the troublemaker at the head of the line. We denote the probability that you get your seat by

$$\mathbf{Prob}_N(A),$$

where A is the event "you get your seat". By our hypothesis, we assume $\mathbf{Prob}_k(A) = \frac{1}{2}$ for $k = 2, 3, \ldots, N-1$. We desire to prove that, assuming our hypothesis is true, then it is also true that $\mathbf{Prob}_N(A) = \frac{1}{2}$.

Now, the crucial observation: Suppose the troublemaker takes the seat belonging to passenger i, where $1 < i < N$. Then passengers $2, 3, \ldots, i-1$ will happily sit in their assigned seats, and passenger i will be the first passenger to choose a seat at random. At this point, we observe that this is the same problem as we started with, but instead of N passengers and seats, the "new" troublemaker (passenger i) has only $N-(i-1)$ seats to choose from. Passenger i is the "new" troublemaker in the following sense: Like the original troublemaker, if i sits in the troublemaker's empty seat, then all remaining passengers will get their seats. Also like the original troublemaker, if i chooses another seat, he will displace a future passenger. Hence, if the original troublemaker takes i's seat, then the probability that you get your seat is $\mathbf{Prob}_{N-i+1}(A)$. We express this compactly with conditional probability, where we define B_i to be the event "troublemaker takes i's seat":

$$\mathbf{Prob}_N(A \mid B_i) = \begin{cases} 1 & i = 1 & \text{(he takes his own seat)} \\ \mathbf{Prob}_{N-i+1}(A) & 1 < i < N & \text{(he takes } i\text{'s seat)} \\ 0 & i = N & \text{(he takes your seat)} \end{cases}$$

Now, we express the probability that you get your seat as the sum of the conditional probabilities:

$$\begin{aligned}
\mathbf{Prob}_N(A) &= \sum_{i=1}^{N} \mathbf{Prob}(B_i)\ \mathbf{Prob}_N(A \mid B_i) \\
&= \sum_{i=1}^{N} \frac{1}{N}\, \mathbf{Prob}_N(A \mid B_i) \\
&= \frac{1}{N}\left(1 + \sum_{i=2}^{N-1} \mathbf{Prob}_{N-i+1}(A) + 0\right) \\
&= \frac{1}{N}\left(1 + \sum_{i=2}^{N-1} \frac{1}{2}\right) \\
&= \frac{1}{N}\left(1 + ((N-1) - 2 + 1)\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

This concludes the proof.

3 Intuitive Interpretation

(Um, so we proved it. Can I explain intuitively why this result makes sense?)